A Global Tracking Control Scheme for Autonomous Vehicles and its Numerical Implementation Using a Variational Integrator

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Abstract

This paper treats the practical and challenging control problem of tracking a prescribed continuous trajectory for an autonomous vehicle in the presence of gravity, buoyancy, fluid dynamic and other forces and moments that are uncertain. These uncertain forces and moments are bounded and may be difficult to model accurately, but they act persistently or over long periods of time. The trajectory is specified in terms of desired attitude and translational motion for a rigid body model of the vehicle. This desired trajectory could be obtained from model-based motion planning schemes. While applications of this control scheme include autonomous aerial and underwater vehicles, we focus on an autonomous underwater vehicle (AUV) application because of its richer, more nonlinearly coupled, dynamics. We use a global characterization of the desired trajectory and trajectory tracking errors in the nonlinear state space. Almost global convergence to the desired trajectory in the nonlinear state space is demonstrated both analytically and through numerical simulations. For numerical simulations of this control scheme, we obtain and use a variational integrator that discretizes the dynamics of the vehicle. This discretization is carried out using discrete variational mechanics, which maintains the Lagrangian structure of the (continuous) dynamics. The discrete dynamics so obtained is implemented as a numerical algorithm to numerically simulate a few AUV maneuvers with varying complexities.

1 Introduction

We consider robust feedback tracking control of an autonomous underwater vehicle (AUV) that is required to track a desired state trajectory in the
presence of uncertainties in the dynamics model. Although the problem of control of AUVs or underwater robots has been extensively studied in the past, our approach to this problem is quite different from previous approaches. We stress the global properties of our control approach, which are important for maneuverable AUVs that do not operate in a cruise mode primarily. We obtain a robust feedback geometric control scheme that ensures practically global asymptotic trajectory tracking in motion states, provided that actuator bounds are not exceeded. To numerically simulate the performance of this control scheme, we also obtain a variational integration scheme that discretizes the continuous dynamics based on discretization of the Lagrange-d’Alembert variational principle of mechanics.

We model an AUV as a rigid body whose motion consists of its translational and rotational motion. This is an accurate model, since an overwhelmingly large percentage of the total energy of the AUV is in the rigid body translational and rotational modes compared to internal (flexible) modes. The dynamics model that we use is based on available models in standard texts on this subject like [1, 2, 3]. The AUV has to track a desired trajectory that can be generated as a translation and attitude time profile that results in a desired state trajectory. The task of transferring the system state under dynamic constraints, from its initial state to a desired final state, is termed motion planning or dynamic interpolation.

Motion planning in nonlinear spaces, like the group of rigid body motions SE(3), under prescribed and well-known dynamics, has been studied extensively in the past. Motion planning for AUVs also has an extensive literature, with application of several control strategies and schemes. Neural net-based controllers for AUVs were reported in [4, 5]. Open-loop geometric control schemes have also been applied to AUVs in [6]-[8], and shown to work well in the absence of model uncertainties. Such open-loop control strategies give trajectories in TSE(3) that transfer the system from the given initial state to the desired final state, while minimizing a cost function that is usually (a combination of) the time taken or control energy expended.

In practice, the presence of dynamic uncertainties like unmodeled external forces and moments, makes it impossible for the desired trajectory to be followed by an open-loop control scheme. A feedback trajectory tracking control scheme, based on feedback of translational and attitude motion states, can be more robust to these dynamic uncertainties. Therefore, we design a feedback trajectory tracking control scheme that ensures that the autonomous vehicle tracks the desired trajectory and is robust to dynamic uncertainties due to its almost global tracking property. Recent research using the framework of geometric mechanics and geometric control has been
successful in demonstrating *almost global* rigid body attitude and angular velocity stabilization and tracking. Such results have been applied to orbiting spacecraft in gravity in [9, 10], to autonomous underwater vehicles in [11] and to simple mechanical systems in [12]. In this paper, we extend these results to tracking control of AUVs with bounded control inputs.

This paper is organized into five sections besides this introduction. Section 2 gives the dynamical model of the AUV used in our theoretical development. Section 3 gives the feedback control approach based on the dynamics model and the trajectory to be tracked. Section 4 derives the discrete equations of motion, which give rise to a variational integrator for the controlled AUV dynamics, using the discrete Lagrange-d’Alembert principle. Section 5 presents numerical simulation results for some feedback tracking maneuvers for the AUV using this variational integrator. Finally, Section 6 presents a concluding discussion of results obtained in this paper and possible future work.

## 2 Model of AUV Dynamics

We treat the AUV dynamics within the framework of geometric mechanics, which makes it convenient to deal with feedback trajectory tracking in a global setting. We denote the position vector of the AUV by \( b = (b_1, b_2, b_3)^T \in \mathbb{R}^3 \), and \( R \in \text{SO}(3) \) is the rotation matrix describing its orientation. Therefore the configuration space of an AUV is the special Euclidean group \( \text{SE}(3) \), the semi-direct product of \( \text{SO}(3) \) and \( \mathbb{R}^3 \), with \((b, R) \in \text{SE}(3)\) denoting the configuration. The state space of an AUV in spatial motion is therefore the tangent bundle \( T\text{SE}(3) \), with the translational and angular velocities in the body-fixed frame denoted by \( \nu = (\nu_1, \nu_2, \nu_3)^T \) and \( \Omega = (\Omega_1, \Omega_2, \Omega_3)^T \) respectively. The kinematic equations for a rigid body are:

\[
\begin{align*}
\dot{b} &= R \nu \\
\dot{R} &= R \Omega^x
\end{align*}
\]  

where the operator \((\cdot)^x : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)\) is defined by \( y^x z = y \times z \), \( \mathfrak{so}(3) \) being the Lie algebra of \( \text{SO}(3) \) or equivalently, the space of skew-symmetric \( 3 \times 3 \) matrices.

Taking the origin of the body-fixed frame for the AUV at its center of gravity \( C_G \), the only moment due to the restoring buoyancy force is the righting moment \(-r_{C_B} \times (\rho g \nu) R^T e_3\), where \( r_{C_B} \) is the vector from \( C_G \) to the center of buoyancy \( C_B \), \( \rho \) is the fluid density, \( g \) the acceleration of gravity, \( \nu \)
the volume of displaced fluid and \( e_3 = [0 \ 0 \ 1]^T \) be the inertial unit vector pointing in the direction of gravity. The dynamics with uncertain forces and moments is given by:

\[
\begin{align*}
M\dot{\nu} &= M\nu \times \Omega + D_\nu(\nu)\nu + (W - \rho g V)R^T e_3 + \varphi_c + \varphi_u, \\
J\dot{\Omega} &= J\Omega \times \Omega + M\nu \times \nu + D_\Omega(\Omega)\Omega \\
&\quad - \tau_{CB} \times (\rho g V)R^T k + \tau_c + \tau_u,
\end{align*}
\]

where \( W \) accounts for the weight of the AUV in air, \( M \) accounts for the mass and added mass, \( J \) accounts for the body moments of inertia and the added moments of inertia. The matrices \( D_\nu(\nu) \) and \( D_\Omega(\Omega) \) represent the drag force and drag momentum, respectively. The vectors \( \varphi_u \in \mathbb{R}^3 \) and \( \tau_u \in \mathbb{R}^3 \) are bounded uncertain force and uncertain moment, respectively, expressed in the body-fixed frame. Finally, \( \varphi_c \) and \( \tau_c \) are the control forces and moments on the AUV, respectively.

Our previous research on the time and energy consumption minimization problem resulted in open loop motion planning for an AUV based on a theoretical model that neglected many factors affecting the experiments. These factors include drag associated with the attached tether, thruster dynamics, underwater currents, etc. This implementation on the testbed AUV was carried out in a controlled environment (a swimming pool, see [6]-[8]). This research is an improvement of our prior research, motivated by the observed failure of such open loop schemes in following a path prescribed by a motion planning algorithm. This research is necessary to translate our AUV experiments from a controlled environment to an ocean environment.

3 Asymptotic Feedback Tracking of AUV motion

Prior literature on local nonlinear control methods for tracking desired attitude motion for a rigid body in the presence of disturbance moments exists, for example [13, 14]. These local methods are not suitable for controlling a highly maneuverable AUV that has to implement large motions. Additionally, it is impossible to obtain a globally asymptotically stable tracking control scheme for the motion of a rigid body or multibody system; as is known in some circles since the early 1980s (see [15] for instance). The first correct treatment of this problem was in [16], which introduced the concept of almost global asymptotic stabilization of such systems by continuous feedback. This was followed by a few correct treatments of this problem, like [17].
The almost global property of the control schemes in \[9, 10\] ensures that a
desired attitude or attitude motion trajectory is tracked starting from almost
any initial state modulo a set of measure zero in the state space. In \[10\], we
also included the effects of additional drag-type disturbance moments (that
are bounded but unknown) on the attitude dynamics, and showed that the
desired trajectory in TSO(3) could be tracked almost globally even in the
presence of such disturbances.

3.1 Trajectory Tracking for AUV

The reference trajectory to be tracked by the AUV can be obtained from an
open loop scheme like that in [6]-[8]. It can be specified in terms of the initial
desired position vector in inertial frame \(b_r(0)\), the initial desired attitude
\(R_r(0)\), and the desired translational and angular velocity time profiles in
body frame, \(\nu^0_r(t)\) and \(\Omega_r(t)\) respectively. The reference trajectory satisfies
the kinematic equation in SE(3):

\[
\dot{g}_r = g_r \zeta_r, \quad g_r = \begin{bmatrix} R_r & b_r \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \zeta_r = \begin{bmatrix} \Omega^x_r \\ 0 \end{bmatrix} + \begin{bmatrix} \nu^0_r \\ 0 \end{bmatrix}.
\]  

(3)

Next we define the trajectory tracking errors, as follows:

\(a(t) \triangleq b(t) - b_r(t)\) = error in inertial position,

\(x(t) \triangleq R^T_r(t)a(t)\) = error in position expressed in reference body frame,

\(Q(t) \triangleq R^T_r(t)R(t)\) = error in body attitude (orientation),

\(\nu(t) \triangleq \nu(t) - Q^T(t)(\nu^0_r(t) + \Omega_r(t)^x x(t))\) = error in body translational velocity,

\(\omega(t) \triangleq \Omega(t) - Q^T(t)\Omega_r(t)\) = error in body angular velocity.

We also define \(\nu_r \triangleq \nu^0_r + \Omega^x_r x\). Thus, we can express the tracking error
kinematics in left-invariant form on SE(3):

\[
\begin{align*}
\dot{x} &= Q\nu \\
Q &= Q\omega^x 
\end{align*} \iff \dot{h} = h\xi,
\]

(4)

where

\[
h = \begin{bmatrix} Q & x \\ 0 & 1 \end{bmatrix} \in \text{SE}(3), \quad \xi = \begin{bmatrix} \omega^x & \nu^0 \\ 0 & 0 \end{bmatrix} \in \text{se}(3).
\]

The dynamics of the AUV can be expressed in terms of the trajectory
tracking errors as follows:

\[
M \dot{v} = M \{\omega \times Q^T \nu_r - Q^T \nu_r\} \\
+ M (v + Q^T \nu_r) \times (\omega + Q^T \Omega_r) + D_\nu(v + Q^T \nu_r) \\
+ \rho g Q^T R^T \nu_r e_3 + \varphi_c, \\
J \dot{\omega} = J (\omega \times Q^T \Omega_r - Q^T \dot{\Omega}_r) - (\omega + Q^T \Omega_r) \times J(\omega) \\
+ Q^T \Omega_r - (v + Q^T \nu_r) \times M(v + Q^T \nu_r) \\
+ D_\Omega(\Omega)(\omega + Q^T \Omega_r) - r_{CB} \times (\rho g) Q^T R^T \nu_r e_3 + \tau_c,
\]

as obtained from equation (2) in the absence of uncertain inputs ($\varphi_u = \tau_u = 0$). The control laws for the inputs $\varphi_c$ and $\tau_c$ are created to asymptotically track the reference trajectory.

### 3.2 Asymptotic Tracking Control Laws in TSE(3)

We design a feedback tracking control scheme, based on Lyapunov-type analysis and full-state feedback, to achieve this task. Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a $C^2$ function that satisfies $\Phi(0) = 0$ and $\Phi'(u) > 0$ for all $u \in \mathbb{R}^+$. Furthermore, let $\Phi'(\cdot) \leq \alpha(\cdot)$ where $\alpha(\cdot)$ is a Class-$K$ function [18]. Let $K$, $L_\nu$, $L_\Omega$ and $N$ be positive definite control gain matrices, with $K = \text{diag}(k_1, k_2, k_3)$ such that $0 < k_1 < k_2 < k_3$. Therefore $\Phi(\text{trace}(K - KQ))$ is a Morse function on SO(3) (where $Q \in \text{SO}(3)$) whose critical points are non-degenerate and hence isolated, according to the Morse lemma [19]. Along the kinematics (4), the time derivative of this function is (see [9] and [10]):

\[
\frac{d}{dt} \Phi(\text{trace}(K - KQ)) = -\Phi'(\text{trace}(K - KQ)) \omega^T \\
\times [k_1 e_1^T Q^T e_1 + k_2 e_2^T Q^T e_2 + k_3 e_3^T Q^T e_3].
\]

We propose the following control laws for $\varphi_c$ and $\tau_c$ to asymptotically track the reference trajectory in TSE(3) in the absence of disturbance inputs:

\[
\varphi_c = -L_\nu v + MQ^T \dot{\nu}_r + (Q^T \Omega_r)^\times (M(v + Q^T \nu_r)) \\
- D_\nu(v + Q^T \nu_r) - \rho g Q^T R^T \nu_r e_3 - Q^T N x, \\
\tau_c = -L_\Omega \omega + JQ^T \dot{\Omega}_r + (Q^T \Omega_r)^\times (JQ^T \Omega_r) \\
+ (Q^T \nu_r)^\times (MQ^T \nu_r) - D_\Omega(\Omega)(\omega + Q^T \Omega_r) \\
+ (\rho g) r_{CB}^\times (Q^T R^T \nu_r e_3) + \Phi'(\text{trace}(K - KQ)) \\
\times [k_1 e_1^T Q^T e_1 + k_2 e_2^T Q^T e_2 + k_3 e_3^T Q^T e_3].
\]
Note that this control law, and hence the trajectories of the closed-loop system, are continuous with respect to the error variables $x, v, Q$ and $\omega$. We next show the almost global asymptotic tracking properties of the closed-loop system (5)-(6) and (8)-(9) when there are no disturbance inputs.

### 3.3 Critical Points for Feedback Attitude Dynamics

Let $\langle \cdot, \cdot \rangle$ denote the trace inner product on the vector space $\mathbb{R}^{m \times n}$, given by

$$\langle A, B \rangle \triangleq \text{trace}(A^T B).$$

We first present a couple of lemmas that are used to prove the main result on asymptotic trajectory tracking.

**Lemma 1.** The function $\Phi(\text{trace}(K - KQ))$ on SO(3) has the set of non-degenerate critical points

$$E_c \triangleq \{I, \ \text{diag}(-1, 1, -1), \ \text{diag}(1, -1, -1),$$

$$\text{diag}(-1, -1, 1)\}.$$ \hspace{1cm} (10)

Further, the unique minimum point of this function is $Q = I \triangleq \text{diag}(1, 1, 1)$.

The proof of this result is given in [10]. A function $\Phi$ of this type has the minimum number of critical points for a Morse function on the nonlinear space SO(3). This lemma is also a corollary of Proposition 1 of [20], which treats Wahba’s problem in attitude determination using similar techniques.

For the closed-loop attitude dynamics of the AUV, we first state a result on the local asymptotic stability of the equilibrium $(I, 0) \in \text{TSO}(3)$. The proof of this result is also provided in [10], which uses a result in [21].

**Lemma 2.** The equilibrium $(I, 0)$ of the closed-loop attitude dynamics given by (6) and the control law (9) is locally asymptotically stable when $(v, \omega) = (0, 0)$. The other equilibria given by $(Q_e, 0)$, where $Q_e \in E_c \setminus \{I\}$, of the closed-loop attitude dynamics under these conditions are unstable. Furthermore, under these conditions the set of all initial conditions converging to the equilibrium $(Q_e, 0)$, where $Q_e \in E_c \setminus \{I\}$ form a lower dimensional manifold.

### 3.4 Asymptotic Convergence Results

We now present our main result on asymptotic convergence of the tracking error dynamics for the closed-loop dynamics (5)-(6) and (8)-(9) to the desired equilibrium $(x_e, Q_e, v_e, \omega_e) = (0, I, 0, 0)$. 
Theorem 1. In the absence of uncertain inputs ($\varphi_u = 0$ and $\tau_u = 0$), the trajectories of the closed-loop tracking error system given by the (5)-(6) and control laws (8)-(9) converge to the set

$$\mathcal{E} = \{(x, Q, v, \omega) \in \text{TSE}(3) : v = 0, \omega = 0, x = 0, \quad Q \in E_c\},$$

where $E_c$ is as defined in (10). Further, the equilibrium $(x_e, Q_e, v_e, \omega_e) = (0, I, 0, 0)$ of the closed-loop system is asymptotically stable in this case and its domain of attraction is almost global.

Proof. For the closed-loop tracking error dynamics given by (5)-(6) and control laws (8)-(9), we propose the following candidate Lyapunov function:

$$V(x, Q, v, \omega) = V_T(x, v) + V_A(Q, \omega),$$

where

$$V_T(x, v) = \frac{1}{2}v^T M v + \frac{1}{2}x^T N x,$$

$$V_A(Q, \omega) = \frac{1}{2}\omega^T J \omega + \Phi(\text{trace}(K - KQ)).$$

Note that $V(x, Q, v, \omega) \geq 0$ and its “attitude component” $V_A(Q, \omega) = 0$ if and only if $(Q, \omega) = (I, 0)$. Thus $V(x, Q, v, \omega)$ is a positive definite function on TSE(3) that is zero only at the desired equilibrium.

We evaluate the time derivative of $V(x, Q, v, \omega)$ along the trajectories of the closed-loop system, using the time derivative of $\Phi(\text{trace}(K - KQ))$ given by equation (7). The time derivative of $V_T(x, v)$ along (5) and (8) is:

$$\dot{V}_T = v^T M \dot{v} + x^T N \dot{x}$$

$$= v^T [M(\omega \times Q^T \nu_r) + (M(v + Q^T \nu_r)) \times \omega - L v].$$

The time derivative of $V_A(Q, \omega)$ along (6), (7) and (9) is:

$$\dot{V}_A = \omega^T J \dot{\omega} - \Phi(\text{trace}(K - KQ)) \omega^T [k_1 e_1^T Q e_1$$

$$+ k_2 e_2^T Q e_2 + k_3 e_3^T Q e_3]$$

$$= -\omega^T [v^T M(v + Q^T \nu_r) + (Q^T \nu_r) \times M v + L \Omega \omega].$$

Therefore, combining (13) and (14) and using the scalar triple product identity, we get

$$\dot{V} = \dot{V}_T + \dot{V}_A = -v^T L v - \omega^T L \Omega \omega \leq 0,$$
and $\dot{V} = 0$ if and only if $v = 0$ and $\omega = 0$.

Recall that $\Phi(\cdot)$ is a strictly increasing monotone function. Hence, for any $(x(0), Q(0), v(0), \omega(0)) \in \text{TSE}(3)$, the set

$$I = \{(x, Q, v, \omega) \in \text{TSE}(3) : V(x, Q, v, \omega) \leq V(x(0), Q(0), v(0), \omega(0))\},$$

is an invariant set of the closed-loop system [9, 10]. By LaSalle’s invariant set theorem, it follows that all solutions that begin in $I$ converge to the largest invariant subset of $\dot{V}^{-1}(0)$ contained in $I$. Since $\dot{V}(x, Q, v, \omega) \equiv 0$ implies $v = \omega \equiv 0$, we substitute this into the closed-loop system equations to get:

$$\dot{V}^{-1}(0) = \{(x, Q, v, \omega) \in \text{TSE}(3) : v \equiv 0, \omega \equiv 0,
\begin{align*}
x &= 0, k_1 e_1^x Q^T e_1 + k_2 e_2^x Q^T e_2 + k_3 e_3^x Q^T e_3 \equiv 0, \\
v &\equiv 0, \omega \equiv 0, \end{align*}\}
= \{(x, Q, v, \omega) \in \text{TSE}(3) : x \equiv 0, Q \in E_c, \\
v \equiv 0, \omega \equiv 0, \}.$$

since $Q^T N x = 0 \Rightarrow x = 0$ as $Q \in \text{SO}(3)$ is invertible and $N$ is positive definite. In this case, each of the four points given by (11) are an equilibrium of the closed-loop dynamics in $\text{TSE}(3)$. Therefore, by LaSalle’s theorem, all solutions of the closed-loop system converge to one of the equilibria in $\mathcal{E} \cap I$, where $\mathcal{E}$ is given by (11).

From Lemma 2, the only stable equilibrium is $(x, Q, v, \omega) = (0, I, 0, 0)$, and all solutions that converge to the other three equilibria form a lower dimensional manifold. Thus, this set of solutions has measure zero in $\text{TSE}(3)$ (see also [9, 10]). Solutions of the closed-loop system that do not start in this manifold, converge asymptotically to the stable equilibrium $(x, Q, v, \omega) = (0, I, 0, 0)$. Therefore, the domain of attraction of this equilibrium is almost global.

4 Discrete Equations of Motion

In this section, we develop a discrete time model for the feedback dynamics of the AUV described in the previous two sections, using a discrete variational approach. The discrete model we obtain is a Lie group variational integrator similar to the ones obtained in [22]-[26]. However, unlike these previous works, our rigid-body dynamics model has non-conservative feedback control torques that are not obtained from an optimal control scheme. We obtain
the feedback control forces and torques using a Lyapunov-type analysis on TSE(3). The Lie group variational integrator for the feedback controlled AUV is derived using the *Lagrange-d’Alembert principle*. Taking variations of the discretization of the action integral leads to the discrete Lagrange-d’Alembert principle.

### 4.1 The Discrete Lagrange-d’Alembert Principle

A variational integrator discretizes *Hamilton’s principle* or Lagrange-d’Alembert principle rather than the continuous equations of motion. For a thorough exposition of the discretization of Hamilton’s principle and Lagrange-d’Alembert principle, we direct our readers to [27]. The discrete version of the Lagrange-d’Alembert principle using generalized coordinates is also given in [28].

Let $U(b, R) : \text{SE}(3) \mapsto \mathbb{R}$ be the potential energy map. The Lagrangian for the motion of the AUV in $\text{SE}(3)$ described in Section 2 is given by

$$
\mathcal{L}(b, v, R, \Omega) = \frac{1}{2} \langle Mv, v \rangle + \frac{1}{2} \langle J\Omega^\times, \Omega^\times \rangle - U(b, R),
$$

where $J$ is a modified inertia matrix defined in terms of the standard moment of inertia matrix $J$ by $J = \text{trace}[J]I - J$ and $I$ is the $3 \times 3$ identity matrix. We first derive the equations using the nonstandard moment of inertia matrix $J$, and then express them in terms of the standard moment of inertia $J$. It can be verified that for any $\Omega \in \mathbb{R}^3$

$$
(J\Omega)^\times = \Omega^\times J + J\Omega^\times.
$$

The continuous Lagrange-d’Alembert principle for the system (2), (8), and (9) is given by

$$
\delta \int_0^T \mathcal{L}(b, v, R, \Omega) dt + \int_0^T (\langle \tau^\times, \Sigma^\times \rangle + \langle \varphi, \delta b \rangle) dt = 0,
$$

where $\tau$ and $\varphi$ denote the external moments and forces, respectively. According to [29], $\Sigma$ is related to $\delta R$ by

$$
\delta R = R\Sigma^\times, \quad \delta \Omega^\times = \Sigma + [\Omega^\times, \Sigma^\times],
$$

where $\Sigma^\times \in \mathfrak{so}(3)$ gives a variation vector field on $\text{SO}(3)$ and $[\ , \ ]$ is the commutator bracket given by $[A, B] = AB - BA$. 
We denote by \( h \neq 0 \) the fixed step size, i.e., \( t_{k+1} - t_k = h \). By \( f_k \) we mean the approximation of \( f \) at time \( t_k \). A discrete Lagrangian \( \mathcal{L}_d \) approximates a segment of the action integral
\[
\mathcal{L}_d(b_k, v_k, R_k, \Omega_k) \approx \int_{t_k}^{t_{k+1}} \mathcal{L}(b, v, R, \Omega) dt.
\]
Similarly we construct \( \mathcal{F} \) to approximate a segment of the virtual work integral
\[
\mathcal{F}_k \approx \int_{t_k}^{t_{k+1}} \left( \langle \tau^x, \Sigma^x \rangle + \langle \varphi, \delta b \rangle \right) dt.
\]
The discrete dynamics is then prescribed by the discrete Lagrange-d’Alembert principle
\[
\delta \sum_{k=0}^{N-1} \mathcal{L}_d(b_k, v_k, R_k, \Omega_k) + \sum_{k=0}^{N-1} \mathcal{F}_k = 0, \tag{16}
\]
where \( \Sigma_0 = \Sigma_N = 0 \) and \( \delta b_0 = \delta b_N = 0 \).

4.1.1 Discretizing the kinematic equations
Let \( R_k \in \text{SO}(3) \) and \( \Omega_k \) denote the attitude and angular velocity, respectively, of the AUV at time \( t_k \). Integrating the kinematic equation \( \dot{R} = R\Omega^x \) gives
\[
R_{k+1} = R_k F_k,
\]
where \( F_k \in \text{SO}(3) \) is given by
\[
F_k = \exp(h\Omega_k^x) \approx I + h\Omega_k^x. \tag{17}
\]
By ensuring that \( F_k \in \text{SO}(3) \), we guarantee that \( R_k \) evolves in \( \text{SO}(3) \). We enforce the skew symmetry on \( \Omega_k^x \) by the following
\[
(J\Omega_k)^x = \Omega_k^x J - J(\Omega^x)^T = \frac{1}{h} \left( (F_k - I)J - J(F_k^T - I) \right)
= \frac{1}{h}(F_kJ - JF_k^T).
\]
From (1) we get that the velocity \( \dot{b}_k \) can be approximated by
\[
\dot{b}_k \approx R_kv_k.
\]
Since $b$ belongs to a linear space we thereby get the approximation

$$b_{k+1} = h R_k v_k + b_k.$$

Hence the discrete counterpart of the kinematic equations (1) becomes

$$\begin{align*}
(J \Omega_k)^\times &= \frac{1}{h} (F_k J - J F_k^T), \\
R_{k+1} &= R_k F_k, \\
b_{k+1} &= h R_k v_k + b_k.
\end{align*}$$

### 4.1.2 Discretizing the dynamic equations

As the discrete Lagrangian we choose the approximation

$$L_d(b_k, v_k, R_k, \Omega_k) = \frac{h}{2} \langle M v_k, v_k \rangle + \frac{h}{2} \langle J \frac{1}{h} (F_k - I), \frac{1}{h} (F_k - I) \rangle$$

$$- \frac{h}{2} \langle U(b_k, R_k) + U(b_{k+1}, R_{k+1}) \rangle$$

For $F$ we choose the approximation

$$F_k = \frac{h}{2} \langle \tau^\times (b_k, v_k, R_k, \Omega_k) + \tau^\times (b_{k+1}, v_k, R_{k+1}, \Omega_k), \Sigma_{k+1}^\times \rangle$$

$$+ \frac{h}{2} \langle \varphi(b_k, v_k, R_k, \Omega_k) + \varphi(b_{k+1}, v_k, R_{k+1}, \Omega_k), \delta b_{k+1} \rangle.$$

For this choice of $L_d$ and $F$ the discrete Lagrange-d’Alembert principle (16) gives

$$\sum_{k=0}^{N-1} \left( \frac{1}{h} \langle \delta F_k, J(F_k - I) \rangle + \langle \delta v_k, M v_k \rangle \\ - \frac{h}{2} \left( \langle \frac{\partial U}{\partial R_k}, \delta R_k \rangle + \langle \frac{\partial U}{\partial R_{k+1}}, \delta R_{k+1} \rangle + \langle \frac{\partial U}{\partial b_k}, \delta b_k \rangle + \langle \frac{\partial U}{\partial b_{k+1}}, \delta b_{k+1} \rangle \right) \\ + \frac{h}{2} \langle \tau^\times (b_k, v_k, R_k, \Omega_k) + \tau^\times (b_{k+1}, v_k, R_{k+1}, \Omega_k), \Sigma_{k+1}^\times \rangle + \frac{h}{2} \langle \varphi(b_k, v_k, R_k, \Omega_k) + \varphi(b_{k+1}, v_k, R_{k+1}, \Omega_k), \delta b_{k+1} \rangle \right) = 0. \tag{21}$$

The variation of $F_k$ can be computed from the definition $F_k = R_k^T R_{k+1},$ using $\delta R_k = R_k \Sigma_k^\times,$ to give

$$\delta F_k = -\Sigma_k^\times F_k + F_k \Sigma_{k+1}^\times.$$
Since the symmetric matrices are orthogononal to the skew-symmetric matrices we get
\[ \langle \delta F_k, J(F_k - I) \rangle = \langle \Sigma_k^x, JF_k^T \rangle - \langle \Sigma_{k+1}^x, F_k^T J \rangle. \]

Using (18) these elements can be rewritten as
\[ \langle \Sigma_k^x, JF_k^T \rangle = \frac{1}{2} \langle \Sigma_k^x, JF_k^T \rangle - \frac{1}{2} \langle \Sigma_{k+1}^x, F_k^T J \rangle = \frac{h}{2} \langle \Sigma_k^x, (J\Omega_k)^x \rangle. \]

The variation of \( \nu_k \) is calculated using (20) as
\[ \delta \nu_k = \delta \left( \frac{1}{h} R_k^T (b_{k+1} - b_k) \right) = \frac{1}{h} R_k^T (\delta b_{k+1} - \delta b_k) - \Sigma_k^x \nu_k. \]

We define \( U(b, R) \in \mathbb{R}^3 \) by
\[ U(b, R) = \frac{\partial U^T}{\partial R} R - R^T \frac{\partial U}{\partial R}, \]

giving
\[ \langle U(b, R), \Sigma^x \rangle = -2 \left( \frac{\partial U}{\partial R} \delta R \right). \]

Using these identities and the end point conditions (21) becomes
\[
\sum_{k=1}^{N-1} \left( \langle \Sigma_k^x, -\frac{1}{2} (J\Omega_k)^x + \frac{1}{2} F_{k-1}^T (J\Omega_{k-1})^x F_{k-1} - \frac{h}{2} (\nu_k \times (M\nu_k))^x \rangle \\
+ \frac{h}{2} \langle \Sigma_k^x, U(b_k, R_k)^x + (\tau^x(b_{k-1}, \nu_{k-1}, R_{k-1}, \Omega_{k-1}) + \tau^x(b_{k-1}, \nu_k, R_{k-1}, \Omega_k)) \rangle \\
+ \langle \delta b_k, R_{k-1} M\nu_{k-1} - R_k M\nu_k - h \frac{\partial U}{\partial b} (b_k, R_k) \rangle \\
+ \frac{h}{2} \langle \delta b_k, \varphi(b_{k-1}, \nu_{k-1}, R_{k-1}, \Omega_{k-1}) + \varphi(b_k, \nu_{k-1}, R_k, \Omega_{k-1}) \rangle \right) = 0.
\]
Using the identity $F \eta \times F^T = (F \eta)^\times$, for all $\eta \in \mathbb{R}^3$ and $F \in SO(3)$, we thus obtain the discrete dynamical equations

$$
M_{uv_{k+1}} = F_k^T M_{v_k} - hR_{k+1}^T \frac{\partial U}{\partial b}(b_{k+1}, R_{k+1}) \\
+ \frac{h}{2} R_{k+1}^T \left( \varphi(b_k, v_k, R_k, \Omega_k) + \varphi(b_{k+1}, v_k, R_{k+1}, \Omega_k) \right),
$$

(22)

$$
J\Omega_{k+1} = F_k^T J\Omega_k + h(M_{v_{k+1}}) \times v_{k+1} + hU(b_{k+1}, R_{k+1}) \\
+ h \left( \tau^\times(b_k, v_k, R_k, \Omega_k) + \tau^\times(b_{k+1}, v_k, R_{k+1}, \Omega_k) \right).
$$

(23)

Together with the discrete kinematics equations (18)-(20) the discrete dynamic equations (22) and (23) constitute the discrete time evolution equations for the system.

4.2 Variational Integrator for AUV Dynamics

5 Numerical Simulation Results

THIS SECTION WILL HAVE RESULTS FOR THE FEEDBACK TRACKING MANEUVERS IN SHASHI’S THESIS

6 Conclusion

CONCLUDING REMARKS

References


